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LATERAL BUCKLING OF THIN-WALLED OPEN MEMBERS WITH SHEAR LAG USING OPTIMIZATION TECHNIQUES

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Abstract—The purpose of this paper is to develop a simple and approximate method with sufficient accuracy for predicting the effect of shearing strains in the middle surface of the walls, which reflect the shear lag phenomenon, on the lateral buckling of thin-walled open member. An energy equation for the lateral buckling has been derived in which the effects of torsion, warping of the member and, especially, the shearing strains in the middle surface of the wall are taken into account. The energy equation can be applied for a prismatic thin-walled member with any kind of open cross-section, for any loading system, and for any end boundary conditions. Some numerical examples by using optimization techniques are given to show the accuracy and applicability of the proposed method here. The results are compared with known results from experiment and finite element method, and good agreement is obtained. Finally, the effect of the shearing strains in the middle surface of the walls on lateral buckling of the thin-walled open member and the effect of the number of Simpson's nodal points taken on optimal values are discussed. © 1997 Elsevier Science Ltd. All rights reserved.

INTRODUCTION

As far as the lateral buckling of the thin-walled member is concerned, the buckling usually occurs by twisting or by a combination of bending and twisting, and the buckling failure will be sensitive to the magnitude of the deflections. Although many recent papers have been dedicated to the lateral buckling of the thin-walled open member, the research seems to be confined to the following assumptions:

(a) the cross-section is rigid in its own plane;
(b) the shearing deformations along the middle surface of the walls are neglected.

Classical analyses were obtained based on the Timoshenko and Gere, and Vlasov's theories (1961). It must be emphasized that the classical analyses are approximate because it neglects the deformation effect of the secondary shearing stresses due to warping restraint which reflect the shear lag phenomenon. Although the problem of shear lag in its manifestations has been recognized for several decades and has been studied in detail both analytically and experimentally for thin-walled closed member, relative few studies have been made on the effect of the shearing strains along the middle surface of the walls on the lateral buckling of thin walled open member. When the shearing strains are taken into account, the mathematical aspect of the problem becomes considerably complicated as it leads, for example, to an integro-differential equation in partial derivatives in the unknown warping function, for which no closed form solution is available (Mentrasti, 1987). A second reason is that Vlasov's assumption (Vlasov, 1961) of no shear deformations along the middle surface of the walls is known to be valid for the thin-walled member of open cross-section. Up to now his theory on the thin-walled open member is taken as the base of many analytical methods.

If the effect of the shearing strains on the buckling is significant, a quick evaluation of the possible shearing strain effect is of importance to a practising engineer at the early stage of the design of thin-walled structures with open cross-section. A computer run at this stage is neither feasible nor economical as even the cross-section itself might be changed in further studies. In this paper, the writer developed a simplified approach to evaluate the effect of

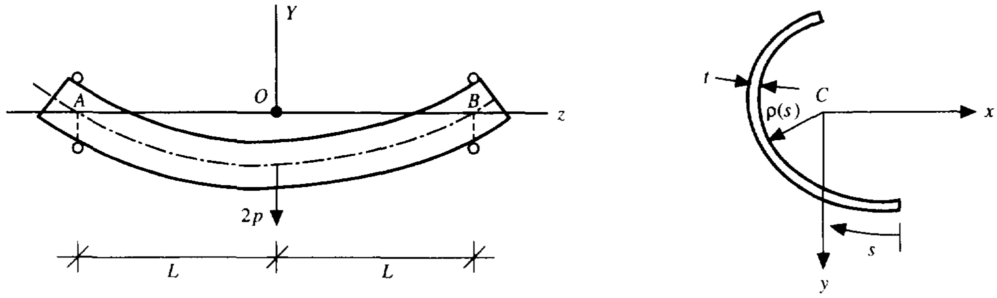

Fig. 1. Thin-walled open member with coordinate system.

the shearing strains in thin-walled open member. To reduce the amount of numerical work in developing the approach, a simply supported doubly symmetric thin-walled member subjected to a lateral point load at midspan, as shown in Fig. 1, is used to illustrate the approach developed to solve for the lateral buckling which the middle surface shear strains are accounted for in the case of open indeformable cross-sections, which arise from the shear stresses in equilibrium with the normal stresses variable along the member axis. An energy equation is derived which governs the equilibrium of the member. The equation is solved by using optimization techniques. The accuracy of the approach is compared with the solutions of experiment and finite element method (the latter did not take into account the shear strains in the middle surface of the walls). Finally, three typical examples are given to show the effect of the shearing strains on the lateral buckling of the thin-walled open member. It is shown that the shearing strains can be neglected in calculating the lateral critical load of the member having a doubly symmetrical cross-section.

ENERGY EQUATION

Consider a prismatic thin-walled open member shown in Fig. 1, the present theory is based on the following three assumptions

(a) The cross-section can be regarded as rigid in its own plane.

According to the Vlasov's assumption, the tangential displacement of any point on the centric line of the thin wall of the cross-section can be expressed as

$$d(s, z) = \rho(s)\phi(z) \tag{1}$$

in which $\rho(s)$ is the distance from the shearing center of the cross-section to the tangential line of the point s, and $\phi(z)$ is the twist angle of the cross-section.

(b) The membrane stresses of σ_x and σ_y parallel to the x and y axes are much smaller than the longitudinal stress σ_z, and thus by Hooke's law, the longitudinal strain

$$\varepsilon_z \approx \sigma_z/E$$

in which E is the Young's modulus of elasticity.

(c) Kollbrunner and Hajdin's assumption for warping displacement is adopted, and thus the distribution of the warping displacement in the thin-walled member can be written as (Kollbrunner and Hajdin)

$$w(s, z) = -\omega(s)\theta(z) \tag{2}$$

in which $\omega(s)$ is the sectorial coordinate with respect to point C; $\theta(z)$ is a function representing the distribution of the warping along the length of the member, and for open cross-section

$$\omega(s) = \int_0^s \rho(s)\,\mathrm{d}s. \tag{3}$$

Longitudinal normal strain

The longitudinal normal strain on the center line of the thin wall due to warping can be obtained by the kinematic equation of elasticity:

$$\varepsilon_w = \partial w/\partial z = -\omega\theta' \tag{4}$$

in which each prime denotes one derivative with respect to z.

The moment of M_x, applied to the major axis, causes the section to twist and, when coupled with shears, causes an additional deflection u in the x-direction during buckling. The moment creates a longitudinal normal strain given by Young and Trahair (1992)

$$\varepsilon_u = -xu''. \tag{5}$$

Shearing strains

Because the shear strains due to bending of the thin-walled member are neglected, the shearing strains of the thin-walled member consist of ones due to warping and uniform torsion. The shearing strains in the middle surface of the walls due to warping are (Vlasov, 1961)

$$\gamma_{sz} = \partial d/\partial z + \partial w/\partial s = \rho\phi' - \partial\omega/\partial s\theta.$$

Substituting eqn (3) into the above equation gives

$$\gamma_{sz} = \rho(\phi' - \theta). \tag{6}$$

When the warping is unrestricted, only the so-called St Venant stresses and strains are present. Generally, the distribution of shearing strains in a thin-walled open member may be shown to be related to the rate of torsion by the expression

$$\gamma_{zs} = 2r\phi' \tag{7}$$

in which r is the distance to any point in the cross-section measured normally from its center line.

Combining eqns (4), (5), (6) and (7) gives

$$\varepsilon = -xu'' - \omega\theta'; \tag{8}$$

$$\gamma = 2r\phi' + \rho(\phi' - \theta). \tag{9}$$

The expression for the strain energy, U, stored in the thin-walled open member is

$$U = \frac{1}{2}\int_0^L \left[\int_{\Sigma s} (E\varepsilon^2 + G\gamma^2)t\,\mathrm{d}s\right]\mathrm{d}z \tag{10}$$

in which Σs is the whole cross-section, and G is the shearing modulus of elasticity.

Substituting eqns (8) and (9) into eqn (10) yields

$$U=\frac{1}{2}\int_0^L\left\{E\left[\left(\int_{\Sigma_s}x^2t\,ds\right)(u'')^2+\left(\int_{\Sigma_s}\omega^2t\,ds\right)(\theta')^2+2\left(\int_{\Sigma_s}x\omega t\,ds\right)u''\theta'\right]\right.$$
$$\left.+G\left[\left(\int_{\Sigma_s}\rho^2t\,ds\right)(\phi'-\theta)^2+\left(\int_{\Sigma_s}4r^2t\,ds\right)(\phi')^2+2\left(\int_{\Sigma_s}2r\rho t\,ds\right)(\phi'-\theta)\phi'\right]\right\}dz. \quad (11)$$

The quantities in parentheses are various geometric properties of the cross-section. In particular:

$I_y=\int_{\Sigma_s}x^2t\,ds$—the second moment of area about the y-axis;
$I_w=\int_{\Sigma_s}\omega^2t\,ds$—the warping moment of inertia;
$J=\int_{\Sigma_s}4r^2t\,ds$—the St Venant torsional constant;
$I_p=\int_{\Sigma_s}\rho^2t\,ds$—the polar moment of inertial of the cross-section about the center of twist.

The other terms are zero for the following reasons:

$\int_{\Sigma_s}x\omega t\,ds=0$ because the warping displacements produce no net moment about the y-axis (Gellin and Lee, 1988);
$\int_{\Sigma_s}2r\rho t\,ds=0$ for a prismatic thin-walled open member.

As a result, eqn (11) reduces as

$$U=\frac{1}{2}\int_0^L[EI_y(u'')^2+EI_w(\theta')^2+GJ(\phi')^2+GI_p(\phi'-\theta)^2]\,dz. \quad (12)$$

The potential energy, V, of the loading system measured from the straight untwisted state is defined by

$$V=-\int_0^L\int_{\Sigma_s}T_i\Delta_it\,ds\,dz \quad (13)$$

in which T_i is a system of conservative surface forces acting on the member in the y–z plane, and Δ_i is the displacement components corresponding to the T_i.

For the lateral buckling analysis, the change in potential energy associated with the buckling process may be expressed as

$$U=-V.$$

Substituting eqns (12) and (13) into the above equations gives

$$\frac{1}{2}\int_0^L[EI_y(u'')^2+EI_w(\theta')^2+GJ(\phi')^2+GI_p(\phi'-\theta)^2]\,dz-\int_0^L\int_{\Sigma_s}T_i\Delta_it\,ds\,dz=0. \quad (14)$$

The present formula of energy equation for lateral buckling can be applied for a prismatic thin-walled member with any kind of open cross-section, for any loading system, and for any end boundary conditions.

OPTIMIZATION PROBLEM AND METHOD OF SOLUTION

The optimum problem is treated here in a form which relates to the common variational method of analysis. For the sake of simplicity, a simply supported open member of length $2L$ having a doubly symmetrical cross-section and subjected to a lateral point load P at midspan is considered. Due to the member symmetry, half the member was analyzed. When P is applied at the centroid of the cross-section, the potential energy, V, is given by Masur and Milbradt (1957)

$$V = -\int_0^L M_x \phi \cdot u'' \, dz \tag{15}$$

in which M_x is the internal moment about the x-axis. For a simply supported beam subjected to a lateral point load at midspan, the potential energy becomes

$$V = -P \int_0^L \phi \cdot u''(L-z) \, dz. \tag{16}$$

Substituting eqn (16) into eqn (14), we have

$$\frac{1}{2}\int_0^L [EI_y(u'')^2 + EI_w(\theta')^2 + GJ(\phi')^2 + GI_p(\phi'-\theta)^2 - 2P\phi u''(L-z)] \, dz = 0. \tag{17}$$

For the case of a member subjected to no lateral forces and with both ends "simply supported", the lateral curvature of u'' in eqn (17) can be eliminated by the following relation

$$EI_y u'' - P(L-z)\phi = 0, \tag{18}$$

to give the following energy expression:

$$\frac{1}{2}\int_0^L [GJ(\phi')^2 + EI_w(\theta')^2 + GI_p(\phi'-\theta)^2 - P^2/EI_y(L-z)^2\phi^2] \, dz = 0. \tag{19}$$

The first variation with respect to ϕ and θ, respectively, yields

$$\int_0^L [GJ\phi' \, \delta\phi' + GI_p(\phi'-\theta) \, \delta\phi' - P^2/EI_y(L-z)^2 \phi \, \delta\phi] \, dz = 0; \tag{20}$$

$$\int_0^L [EI_w \theta' \, \delta\theta' - GI_p(\phi'-\theta) \, \delta\theta] \, dz = 0. \tag{21}$$

Noting that the processes of variation and differential can be permutable, eqns (20) and (21), respectively, can be integrated by parts as follows:

$$[GJ\phi' \, \delta\phi + GI_p(\phi'-\theta)] \, \delta\phi \Big|_0^L - \int_0^L [GJ\phi'' + GI_p(\phi'-\theta)' + P^2/EI_y(L-z)^2\phi] \, \delta\phi \, dz = 0; \tag{22}$$

$$EI_w \theta' \, \delta\theta \Big|_0^L - \int_0^L [EI_w \theta'' + GI_p(\phi'-\theta)] \, \delta\theta \, dz = 0. \tag{23}$$

To satisfy the extremum condition of variation for any arbitrary values of $\delta\phi$ and $\delta\theta$, the terms under the integral must vanish. This condition produces the following governing differential equations for the lateral buckling:

$$GJ\phi'' + GI_p(\phi'-\theta)' + P^2(L-z)^2/EI_y\phi = 0; \tag{24}$$

$$EI_w \theta'' + GI_p(\phi'-\theta) = 0. \tag{25}$$

The corresponding natural boundary conditions at $z = 0$ and $z = L$, respectively, must be satisfied too

$$GJ\phi' + GI_p(\phi' - \theta) = 0; \tag{26}$$

or $\phi = \text{Const.}$;

$$EI_w\theta' = 0; \tag{27}$$

or $\theta = \text{Const.}$

Equation (24) gives

$$\theta' = \mu\phi'' + P^2/(EI_y GI_p)(L-z)^2\phi. \tag{28}$$

Differentiating with respect to z gives

$$\theta'' = \mu\phi''' + P^2[-2(L-z)\phi + (L-z)^2\phi']/(EI_y GI_p). \tag{29}$$

From eqn (25) we have

$$\begin{aligned} \phi' - \theta &= -EI_w\theta''/GI_p \\ &= -EI_w\{\mu\phi''' + P^2[-2(L-z)\phi + (L-z)^2\phi']/(EI_y GI_p)\}/GI_p \end{aligned} \tag{30}$$

in which $\mu = 1 + J/I_p$.

Substituting eqns (28) and (30) into eqn (19) and introducing

$$\xi = (L-z)/L; \lambda = p/p_c; \quad \text{and} \quad (p_c)^2 = (EI_w EI_y)/L^6$$

give

$$\begin{aligned} &\left\{\int [\alpha^4/L^4\xi^4\phi^2 + \int \alpha^6/L^6(2\xi\phi + \xi^2\phi')^2]\,\mathrm{d}\xi\right\}\lambda^4 \\ &\quad + \left\{\int [2\mu\alpha^2/L^2\xi^2\phi\phi'' + 2\mu\alpha^4/L^4(2\xi\phi + \xi^2\phi')\phi''' - \xi^2\phi^2]\,\mathrm{d}\xi\right\}\lambda^2 \\ &\quad + \int [\mu^2\alpha^2/L^2(\phi''')^2 + \mu^2(\phi'')^2 + k^2L^2(\phi')^2]\,\mathrm{d}\xi = 0 \end{aligned} \tag{31}$$

in which $k^2 = GJ/EI_w$; and $\alpha^2 = EI_w/GI_p$.

When the member buckles laterally, the smallest such value that yields a non-trivial second solution for the member is known as the critical load called P_{cr}. The critical value of P_{cr} may be obtained by minimizing eqn (31) with respect to the angle of twist ϕ. In mathematical terms, the optimization problem may be cast in the following form:

Find the critical load P_{cr}.
Minimize eqn (31).
Subject to

$$\phi(0) = 0; \tag{32a}$$

$$\phi''(0) = 0; \tag{32b}$$

$$\phi'(1) = 0; \tag{32c}$$

$$\mu\Phi'''(1) - k^2L^2\phi'(1) = \alpha^2/L^2\lambda^2[2\phi(1) + \phi'(1)]. \tag{32d}$$

For a cantilever thin-walled member subjected to a lateral point load, P, at the free end

$$V = -P\int_0^L \phi \cdot u''z\,dz. \tag{33}$$

When a thin-walled member under equal and opposite end moments, M, and V is given by

$$V = -\int_0^L M\phi \cdot u''\,dz, \tag{34}$$

substituting eqns (33) and (34) into eqn (14), respectively, and repeating the derivation procedure outlined above, we can develop the formula corresponding to the cantilever boundary under pure moment, respectively, for lateral buckling analysis of thin-walled open member by using the optimization technique.

To solve the optimization problem, ϕ is approximated by a suitable function that satisfies the boundary conditions pertaining to the type of problem. In general, for lateral buckling of a thin-walled member, a solution of the optimization problem can be expanded in the Fourier series as

$$\phi(\xi) = \sum_i \sin i\pi\xi/2 \quad i = 1, 2, \ldots, m. \tag{35}$$

The integrations in eqn (31) are evaluated numerically using Simpson's rule. A numerical optimization technique based on the direct search optimization method (Gallagher and Zienkiewicz, 1973) is used for the purpose of minimization of eqn (31). The Simpson's nodal points taken in the approximation of ϕ are equally spaced along the length of the member, and the effect of the number of the nodal points taken on the optimal results will be discussed later.

NUMERICAL EXAMPLES

Case 1. Simply supported thin-walled member

Because thin-walled I-beams are traditionally used extensively in buildings and bridges, for the sake of brevity, only two doubly symmetric simply supported thin-walled I-members subjected to a lateral point load at midspan are selected as numerical examples. The data for these examples come from Thevendran and Shanmugam (1991), and Away *et al.* (1978). The results considering the shearing strains in the middle surface of the walls in this paper are compared with the experimental result and more exact value obtained by using the finite element method as shown in Table 1 in which five equally spaced nodal points are used.

It can be seen from Table 1 that the proposed lateral buckling loads in this paper are all close to the experimental one and the one determined by finite element method, the later did not take into account the shearing strains. Their relative differences to the present method are only 3.78% and −2.47%, respectively.

Table 1. Comparison of results of P_{cr}

Cases	Example of Thevendran *et al.*		Example of Away *et al.*	
Methods	Experiment	Present M	F.E.M. 2 × 10 mesh	Present M
P_{cr}	707.80 N	681.05 N	73.3 kips	75.11 kips
Difference	3.78%		−2.47%	

Table 2. Comparison of results of P_{cr}

Case	Methods	P_{cr} (N)
1	Experiment of Thevendran *et al.*	681.10
2	Present analysis	704.77

Table 3. Comparison of results of M_{cr}

Case	Methods	M_{cr} (kN-m)
1	Example given by Xia and Pan	431.00
2	Present analysis	431.15

Case 2. *Cantilever thin-walled member*

A test was done by Thevendran *et al.* (1991), his specimen is cantilever *I*-beam of length $L = 465$ mm, web thickness $t_w = 6$ mm, overall web depth $d = 75$ mm, flange thickness $t_f = 10$ mm, and flange width $b = 23.5$ mm. The test is carried out using a specimen from plexiglass sheets having average values of Young's modulus $E = 2860$ N/mm^2 and Poisson's ratio $\nu = 0.36$. The present result is compared with the one obtained from the experiment in Table 2.

From Table 2 it can be seen that the P_{cr} value obtained from the present method agrees well with the result from the experiment, the relative difference is within 3.48%.

Case 3. *Under pure moment*

The example came from China (Xia and Pan, 1988). An *I*-beam, subjected to equal and opposite end moments, was analyzed. The related data are as follows: $E = 2.1 \times 10^5$ MPa; $G = 0.81 \times 10^5$ MPa; $I_y = 3549$ cm^4, $J = 131$ cm^4; $I_w = 1{,}717{,}716$ cm^6; $2L = 8.0$ m. The comparisons of the results obtained are summarized in Table 3.

It can be seen that the result from this paper is very close to the analytic one from Xia and Pan, and the relative difference is almost nil.

EFFECT OF SHEARING STRAINS ON LATERAL BUCKLING

If the shear strains along the middle surface of the walls are neglected, eqn (31) will be reduced to

$$\left(\int \xi^2 \phi^2 \, d\xi\right)\lambda^2 = \int [(\phi'')^2 + k^2 L^2 (\phi')^2] \, d\xi. \tag{36}$$

The λ can be solved by the same calculation as the previous section. The same examples as the above section are selected to demonstrate the effect. The different results considering the shearing strains in the middle surface of the walls are compared with the ones neglecting the shearing strains in Table 4 in which five equally spaced Simpson's nodal points are used.

From the examples as shown in Table 4, it can be seen that the effect of the shearing strains in the middle surface of the walls on the lateral buckling is small. Although this comparison cannot fully exhaust the comparative analysis owing to the fact that the shearing strains are considered directly only in the present formula, the conclusion should

Table 4. Effect of shearing strains on values of P_{cr}

Cases	Example of Thevendran *et al.*	Example of Away *et al.*
γ_{zs} is neglected	681.0773 N	75.16847 kips
γ_{zs} is considered	681.0501 N	75.11375 kips
Differences	0.01%	0.07%

Table 5. Effect of number of the nodal points taken on P_{cr}

Cases	Example 1	Example 2
$n = 2$	681.4723 N	75.16751 kips
$n = 3$	680.7152 N	75.08400 kips
$n = 4$	680.5875 N	75.07101 kips
$n = 5$	680.5660 N	75.06755 kips
$n = 6$	680.5551 N	75.06634 kips

$n+1$ is the number of Simpson's nodal points taken.

be accepted because, for the case of open cross-section, the ratio of the St Venant torsional rigidity GJ to the warping rigidity EI_w over the cross-section is low. In comparison with the primary St Venant shear stresses, the secondary shear stresses due to the shearing strains in the middle surface of the walls should be small, so that their deformation effect may be neglected. On the other hand, for a thin-walled closed member, their effects on torsional equilibrium are not negligible because of the large lever arms with which they act.

EFFECT OF NUMBER OF SIMPSON'S NODAL POINTS TAKEN ON RESULTS

If we divided the interval (a, b) into n strips, where n is even, we can write

$$\int_a^b f(z)\,dz \approx (b-a)/3(f_0+4f_1+2f_2+4f_3+2f_4+\cdots+2f_{n-2}+4f_{n-1}+f_n).$$

This is Simpson's rule, whose error is approximately (Scraton, 1984)

$$E_n(f) = -(b-a)^5 f''''/(180n^4). \quad (37)$$

The relative error between n and $n-1$ strips is taken as

$$E_n(f)/E_{n-1}(f) = [(n-1)/n]^4. \quad (38)$$

It is found from eqn (38) that taking too many numbers is futile because when $n \to \infty$, the relative error approximates to 1. To select a suitable n, the values of the lateral buckling load are examined by taking different numbers of Simpson's nodal points. The differences in the results due to the five different numbers taken in the course of numerical integration as shown in Table 5 almost disappear when n = 3–6. With four strips we have virtually got machine accuracy.

CONCLUSIONS

From the investigations in this paper, the following conclusions can be drawn.

(a) An energy equation for the lateral buckling of thin-walled open member has been derived in which the effects of torsion, warping especially, and the shearing strains in the middle surface of the walls are taken into account. The equation can be applied for prismatic thin-walled member with any kind of open cross-section, for any loading system, and for end boundary conditions.

(b) A numerical analysis for the lateral buckling of doubly symmetrical simply supported thin-walled open member by using optimization technique has been presented in this paper. In comparison with the known results from experiment and finite element method, a satisfactory agreement has been obtained here.

(c) The same method can be applied for other types of load and end constraint conditions.

(d) The effect of the shearing strains in the middle surface of the walls on lateral buckling of simply supported thin-walled *I*-member having double symmetrical cross-section is small.

(c) In the course of numerical integration, taking too many numbers of Simpson's nodal point is futile.

REFERENCES

Away, H. U., Jonson, C. P. and Will, A. M. (1978) Lateral and local buckling of beams and frames. *Journal of the Structural Division ASCE* **103**(ST9), 1821–1832.

Gallagher, R. H. and Zienkiewicz, O. C. (1973) *Optimum Structural Design: Theory and Applications*. John Wiley, New York.

Gellin, S. and Lee, G. C. (1988) Finite elements available for the analysis of non-curved thin-walled structures. In *Finite Element Analysis of Thin-Walled Structures*, pp. 1–30. Elsevier Applied Science, London.

Kollbrunner, C. F. and Hajdin, H. Wöibkraft-Torsion Dünnwardiger Stäbe mit Gesehlossenem Profil. Schweizer Stahlbauverband Mitteilungen der Technischen Kommision, Heft 32.

Masur, E. F. and Milbradt, K. P. (1957) Collapse strength of redundant beams after lateral buckling. *Journal of Applied Mechanics, Transactions, ASME* **24**(2), 283–288.

Mentrasti, L. (1987) Torsion of closed cross-section thin-walled beams: the influence of shearing strain. *Journal. of Thin-walled Structures* **5**, 277–305.

Scraton, R. E. (1984) *Basic Numerical Methods (An Introduction to Numerical Mathematics on a Microcomputer)*. Edward Arnold, Baltimore, Maryland, U.S.A.

Thevendran, V. and Shanmugam, N. E. (1991) Lateral buckling of doubly symmetric beams containing openings. *Journal of Engineering Mechanics, ASCE* **117**(7), 1427–1441.

Timoshenko, S. P. and Gere, J. M. (1961) *Theory of Elastic Stability*. 2nd edn. McGraw Hill, New York.

Vlasov, V. Z. (1961) *Thin-Walled Elastic Beams*. 2nd edn. Israel Program for Scientific Translations, Jerusalem, Israel.

Xia, Zhibing and Pan, Aching (1988) *Theory of Structural Stability*. 2nd end. Advanced Education Press, Beijing, China (in Chinese).

Young, L. P. and Trahair, N. S. (1992) Energy equation for beam lateral buckling. *Journal of Structural Engineering, ASCE* **118**(6), 1462–1479.